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Equivalent linearization method using Gaussian mixture (GM-ELM) for nonlinear random vibration analysis

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ABSTRACT

A new equivalent linearization method is developed for nonlinear random vibration analysis. The method employs a Gaussian mixture distribution model to approximate the probabilistic distribution of a nonlinear system response. The parameters of the Gaussian mixture model are estimated by an optimization algorithm which requires a few rounds of dynamic analysis of the nonlinear system. Due to properties of the Gaussian mixture distribution model, the proposed Gaussian mixture based equivalent linearization method (GM-ELM) can decompose the non-Gaussian response of a nonlinear system into multiple Gaussian responses of linear single-degree-of-freedom oscillators. Using a probabilistic combination technique, the linear system of GM-ELM can provide the response probability distribution equal to the Gaussian mixture estimation of the nonlinear response distribution. Using the linear system of GM-ELM in conjunction with linear random vibration theories, response statistics such as the mean up-crossing rate and first-passage probability of the nonlinear system can be conveniently computed. In order to facilitate applications of GM-ELM in earthquake engineering practice, a response spectrum formula is also proposed to compute the mean peak response of the nonlinear system by using the elastic response spectra representing the peak responses of the linear single-degree-of-freedom oscillators. Finally, two numerical examples are presented to illustrate and test GM-ELM. The analysis results obtained from GM-ELM are compared with those obtained from the conventional ELM and Monte-Carlo simulation. The supporting source code and data are available for download at https://github.com/ziqidwang/GitHub-GM-ELM-code.git.

1. Introduction

Random vibration analysis of structural systems subjected to stochastic excitations, such as earthquake, wind or wave loading, has been the focus of numerous research efforts in the past several decades. Fundamental progress has been made in analyzing linear structures, yet the analysis of general multi-degree-of-freedom (MDOF) nonlinear systems still poses significant challenges. To tackle challenges of nonlinear random vibration analysis, various approaches have been developed, among which the equivalent linearization method (ELM) [1–4] has gained wide popularity due to its applicability to general MDOF nonlinear systems. Other classical methods, Fokker-Planck equation, stochastic averaging, moment closure and perturbation for example (see [5] for a review of these methods), although probably more accurate, are mainly restricted to specific (and usually simple) nonlinear systems. Sampling based methods [6] have fewer restrictions, but they are computationally expensive, especially for complex MDOF structural systems with a low level of failure probability.

In ELM, the nonlinear system of interest is replaced by an equivalent linear system with the same DOF. In a conventional approach of ELM, parameters of the equivalent linear system are determined via minimizing the mean-square error between the responses of the nonlinear and linear systems [2,3]. ELM approaches based on various discrepancy measures between the two systems have been considered [4,7], yet they are not as widely used as the conventional approach. In general, ELM could be accurate in estimating the mean-square responses, but it may not capture the non-Gaussianity of the nonlinear responses effectively. As a consequence, using ELM to estimate response statistics such as response probability distributions could be far from correct, especially in the tail region of the distribution. As an alternative to the conventional ELM, a non-parametric ELM based on first-order reliability method (FORM) [8,9], namely the tail-equivalent linearization method (TELM), has been proposed recently [10]. In TELM, an equivalent linear system is numerically obtained in terms of a discretized impulse-response function or frequency-response function, using

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knowledge of the 'design point' determined from FORM analysis. In comparison to the conventional ELM, TELM has superior accuracy in estimating the response probability distributions, especially in the tail region. At the same time, however, TELM is computationally more expensive than the conventional ELM, since TELM involves performing FORM analysis for a sequence of response threshold values. Moreover, in contrast to ELM, TELM is restricted to structural models using smooth constitutive laws and weak stiffening systems.

In this paper, a new equivalent linearization method is developed for nonlinear random vibration analysis. The method employs a Gaussian mixture (GM) [11] distribution model to approximate the probabilistic distribution of a nonlinear system response. The parameters of the GM model are estimated by an optimization algorithm which requires a few rounds of dynamic analysis of the nonlinear system. Due to properties of the GM distribution model, the proposed GM based equivalent linearization method (GM-ELM) can decompose the non-Gaussian response of a nonlinear system into multiple Gaussian responses of linear single-degree-of-freedom (SDOF) oscillators. Using a probabilistic combination technique, the linear system of GM-ELM can provide the response probability distribution equal to the Gaussian mixture estimation of the nonlinear response distribution.

Similar to the conventional ELM, GM-ELM identify parameters of the linear SDOF oscillators using those of the GM model and the original nonlinear system. In contrast to ELM, GM-ELM can capture the non-Gaussianity of the nonlinear responses, and it has superior accuracy in estimating response statistics such as the mean up-crossing rate, the maximum response distribution and the mean peak response. On the other hand, in comparison with TELM, GM-ELM does not involve reliability analysis requiring response gradient/sensitivity computation, which may require smooth constitutive laws in structural models.

The paper starts with an introduction of main theories of the proposed GM-ELM, followed by computational details of the method. Then, the application of GM-ELM in general random vibration analysis, e.g. estimating the mean square response, mean up-crossing rate and first-passage probability, is introduced. In particular, to facilitate practical applications of GM-ELM in earthquake engineering, a response spectrum formula is developed to compute the mean peak response of the nonlinear system by use of existing linear elastic response spectra, i.e. without developing a nonlinear/inelastic response spectrum for each class of system. Finally, two numerical examples are presented to illustrate and test GM-ELM. The first example studies a cubic SDOF oscillator subjected to white noise excitation. The second example investigates a 6-DOF shear-building model with bilinear hysteretic force-deformation relations for each story, subjected to a stochastic ground motion described by a modified Kanai-Tajimi model. Throughout the paper, analysis results obtained from GM-ELM are compared with those obtained from the conventional ELM and Monte-Carlo simulation. The primary focus of the paper is to provide the theoretical framework of GM-ELM. Therefore, a thorough comparison between TELM and GM-ELM will be provided in upcoming papers, in which specialized knowledge of structural reliability (FORM analysis in particular) involved in TELM needs to be presented in detail.

2. Main theories of the equivalent linearization method using Gaussian mixture (GM-ELM)

2.1. Basic procedures of GM-ELM

The proposed GM-ELM establishes a set of linear oscillators through a Gaussian mixture (GM) representation of the probability density function (PDF) for a generic nonlinear response of interest, and a physical interpretation of the GM model. The GM model is employed because: a) each Gaussian density in the GM model can be naturally related to a linear system, since the response of a linear system to Gaussian excitation is also Gaussian, and b) despite its simplicity, GM [11] can represent general probability densities that show complex shapes, especially for dynamic responses of nonlinear structures [2]. Therefore, the GM model enables one to describe a nonlinear system subjected to Gaussian excitations by a set of linear systems.

The PDF of a GM model is expressed by

$$p_{GM}(z;\mathbf{v}) = \sum_{k=1}^{K}\alpha_k f_{\mathcal{N}}(z;\mu_k,\sigma_k) \tag{1}$$

in which K denotes the number of Gaussian densities in the mixture, $\alpha_k,\ k=1,\ldots,K$, are relative weights of the Gaussian densities satisfying $\sum_{k=1}^{K}\alpha_k = 1$ and $\alpha_k > 0$ for $\forall k$, and $f_{\mathcal{N}}(z;\mu_k,\sigma_k)$ denotes the Gaussian PDF with mean μ_k and standard deviation σ_k. Thus, the distribution parameters of the GM model are summarized as $\mathbf{v} = \{\alpha_1,\ldots,\alpha_K,\mu_1,\ldots,\mu_K,\sigma_1,\ldots,\sigma_K\}$.

The basic procedures of GM-ELM are described as follows.

Step 1. Use a GM model to represent the PDF of the nonlinear system response of interest.
Step 2. Identify a set of linear oscillators using parameters of the GM model in conjunction with parameters of the original nonlinear system.
Step 3. Use the linear system obtained from Step 2 to compute response statistics of interest.

The primary focus of this paper is on the development of Step 2 and Step 3. For Step 1, if the PDF of a nonlinear response is not available, a simulation based approach will be employed for illustrative purpose.

2.2. Linear system associated with the Gaussian mixture

Suppose the probability distribution of a nonlinear response is represented by a GM model with K densities. From Eq. (1), the random nonlinear response Z can be described as

$$Z \cong \sum_{k=1}^{K} I_k \cdot Z_k = \sum_{k=1}^{K} I_k \cdot (\mu_k + D_k) \tag{2}$$

where I_k is the kth element of a K-dimensional random vector in which only one element takes 1 while the others take 0 according to the probabilities $\alpha_k, k=1,\ldots,K$, with $\sum_{k=1}^{K}\alpha_k = 1$, while Z_k follows Gaussian distribution with mean μ_k and standard deviation σ_k. In Eq. (2), Z_k is alternatively described as $\mu_k + D_k$ in which D_k is a random variable whose distribution is a zero-mean Gaussian distribution with standard deviation σ_k. The stationarity assumption of the response will be used throughout the paper, thus α_k, μ_k and σ_k are independent of time. The terms Z, I_k, Z_k and D_k in Eq. (2) do vary with time, but the notation has been simplified to omit this detail.

Eq. (2) depicts a *probabilistic decomposition* of a non-Gaussian response into multiple Gaussian responses, which is analogous to the modal analysis approach. In the modal analysis approach, the response of a linear MDOF system is represented by multiple linear single-degree-of-freedom (SDOF) oscillators. In the proposed approach, the non-Gaussian response of a nonlinear system is represented by multiple linear oscillators whose relative importance (in a probabilistic sense), 'location' (with respect to the origin of the z-axis, see Fig. 1), and root-mean-square oscillation around

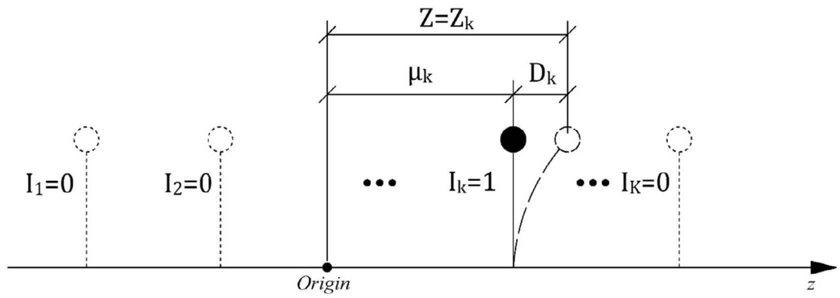

Fig. 1. A physical interpretation of densities in the identified Gaussian mixture model.

the specified location are represented respectively by α_k, μ_k and σ_k, $k = 1, \ldots, K$. This concept is illustrated in Fig. 1.

Despite the similarities, the aforementioned concept is still fundamentally different from the conventional modal decomposition in the sense that at a specified time point, only one linear system/-mode could be selected to represent the nonlinear system (due to the fact that only one I_k could take the value 1 in Eq. (2)). The deformation D_k of the linear system is considered as the response to the zero-mean component of the stochastic excitation, so that D_k is a zero-mean process. As shown in Eq. (2), the zero-mean condition of D_k assures the response probability distribution of the linear system follows the GM model.

From the linear system in Eq. (2), the conditional CDF of response Z can be written as

$$\Pr[Z < z | I_k = 1] = \Pr[Z_k < z] = \Pr[D_k < z - \mu_k] = \Phi\left(\frac{z - \mu_k}{\sigma_k}\right) \tag{3}$$

in which $\Phi(\cdot)$ is the CDF of the standard normal distribution. Using Eq. (3) and the law of total probability, the CDF of Z can be written as

$$\Pr[Z < z] = \sum_{k=1}^{K} \Pr[I_k = 1] \Pr[Z < z | I_k = 1] = \sum_{k=1}^{K} \alpha_k \Phi\left(\frac{z - \mu_k}{\sigma_k}\right) \tag{4}$$

Starting from Eq. (2), the mean of Z is derived as

$$E[Z] = \sum_{k=1}^{K} \alpha_k E[D_k + \mu_k] = \sum_{k=1}^{K} \alpha_k \mu_k \tag{5}$$

Next, starting from Eq. (2) and using Eq. (5), the variance of Z is derived as

$$\begin{aligned} E\left[(Z - E[Z])^2\right] &= \sum_{k=1}^{K} \alpha_k E\left[(D_k + \mu_k)^2\right] - \left(\sum_{k=1}^{K} \alpha_k \mu_k\right)^2 \\ &= \sum_{k=1}^{K} \alpha_k (\sigma_k^2 + \mu_k^2) - \sum_{k=1}^{K} \alpha_k \left(\sum_{k=1}^{K} \alpha_k \mu_k\right)^2 \\ &= \sum_{k=1}^{K} \alpha_k \left[\sigma_k^2 + \mu_k^2 - \left(\sum_{k=1}^{K} \alpha_k \mu_k\right)^2\right] \end{aligned} \tag{6}$$

In terms of the aforementioned derivation, it is seen that the mixture of linear systems established in GM-ELM can provide response statistics equal to the Gaussian mixture estimation of the nonlinear response statistics. It should be noted that the current section provides the idea of how a linear-system 'concept' can be established from a Gaussian mixture structural response PDF model. Besides the GM model, the specific parameters of the linear system require some additional information on the structural properties. The determination of specific parameters of the linear system will be described in the following sections.

To provide a preliminary idea on the effectiveness of the proposed GM-ELM approach, Fig. 2 shows the GM-ELM estimation for the PDF and complementary cumulative distribution function (CDF) of a cubic SDOF oscillator response [2], compared with the solution by the conventional ELM and the exact solution obtained from Fokker-Planck equation. Also a Gaussian distribution with the variance equal to the variance of the nonlinear response is illustrated in Fig. 2, to indicate the non-Gaussianity of the nonlinear response. Details of this example and estimations on other response statistics will be described in Section 6.

3. Details of GM-ELM

3.1. Identifying optimal parameters of the Gaussian mixture

GM-ELM requires knowledge of the PDF of a nonlinear response. Just to show how this method can be used even when the PDF of the nonlinear response is not available, a simulation based approach is employed in this paper for illustrative purpose. Note that other non-simulation based PDF estimation approaches are available in the literature [12–14]. The framework of GM-ELM developed in this paper is independent of how the PDF is estimated, thus one could explore the usage of other PDF estimation approaches for GM-ELM.

Given a set of samples z_i drawn from nonlinear dynamic analysis or directly from the nonlinear response PDF (if the response PDF is known), the 'best' GM that fits the observed data can be obtained from [15]

$$\mathbf{v}^* \cong \arg\max_{\mathbf{v}} \frac{1}{N} \sum_{i=1}^{N} \ln p_{GM}(z_i; \mathbf{v}) \tag{7}$$

in which arg max denotes the argument of the maxima. Note that if the nonlinear response PDF is unknown, each sample point z_i involves a dynamic analysis, therefore Eq. (7) could be computationally demanding, given that a large sample size N may be required to have an accurate estimation of $\mathbf{v}^*$. However, the computational demand of Eq. (7) could be significantly reduced by assuming the response process $Z(t)$ is stationary ergodic, given the input excitation is stationary. This is because, for a stationary ergodic process, the response at every time point could be used as the sample point z_i in Eq. (7), and consequently by only performing a few rounds of dynamic analysis one would acquire a large set of sample points.

The optimization expressed by Eq. (7) can be solved by an iterative rule to update parameters of the GM model (see, e.g. [11])

$$\mu_k = \frac{\sum_{i=1}^{N} \Upsilon_{i,k} z_i}{\sum_{i=1}^{N} \Upsilon_{i,k}} \tag{8}$$

$$\sigma_k = \frac{\sum_{i=1}^{N} \Upsilon_{i,k} (z_i - \mu_k)^2}{\sum_{i=1}^{N} \Upsilon_{i,k}} \tag{9}$$

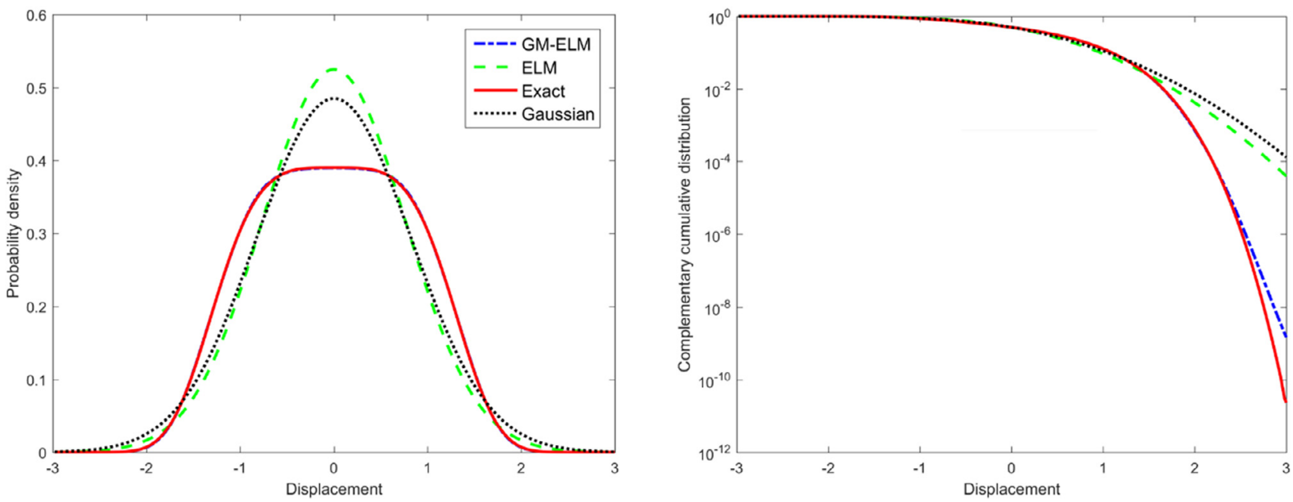

Fig. 2. PDF and complementary CDF estimations of a cubic SDOF oscillator by GM-ELM and conventional ELM.

$$\alpha_k = \frac{\sum_{i=1}^{N} \Upsilon_{i,k}}{N} \tag{10}$$

in which coefficient $\Upsilon_{i,k}$ is expressed as

$$\Upsilon_{i,k} = \frac{\alpha_k f_{\mathcal{N}}(z_i; \mu_k, \sigma_k)}{\sum_{j=1}^{K} \alpha_j f_{\mathcal{N}}(z_i; \mu_k, \sigma_j)} \tag{11}$$

Note that $\Upsilon_{i,k}$ represents the posterior probability that the kth Gaussian density in the mixture is picked for a given z_i. The optimal GM parameters can be identified via the following algorithm:

1) *Obtaining sample points:* a) perform a single run of dynamic analysis of the structure subjected to a random realization of the stochastic excitation, record the response history and compute the standard deviation of the response history, denoted as $\sigma^{(1)}$; b) repeat step a) so that a sequence $\sigma^{(1)}, \ldots, \sigma^{(M)}$ associated with the standard deviations of the sequence of response histories is obtained, and stop the iteration if $\frac{\text{std}(\sigma^{(1)},\ldots,\sigma^{(M)})}{\sqrt{M}\cdot\text{mean}(\sigma^{(1)},\ldots,\sigma^{(M)})} < Tol$ is satisfied, where $\text{std}(\sigma^{(1)}, \ldots, \sigma^{(M)})$ and $\text{mean}(\sigma^{(1)}, \ldots, \sigma^{(M)})$ respectively denote the sample standard deviation and sample mean of the sequence $\sigma^{(1)}, \ldots, \sigma^{(M)}$. For each of the M response histories, the response values at $\bar{N}$ time points are selected (see *Remark 1* below) as the sample points to estimate the GM model, thus the total sample size is $N = M \cdot \bar{N}$.
2) *Initializing GM model:* Set value K (see *Remark 2* below), which denotes the number of Gaussian distributions in the mixture. Set other initial parameters of the mixture $p_{GM}(z; \mathbf{v})$ as well. For example, one can set all α_k to $1/K$, μ_k to a random number drawn from a uniform distribution covering the domain in which the response values are of interest, and all σ_k to 1.
3) *Updating:* Use Eqs. (8)–(11) in the order (11), (8)–(10) iteratively to update parameters of the GM. Stop the updating process if $|ce^{(s)} - ce^{(s-1)}|/ce^{(s)} \leqslant Tol$ is satisfied, where $ce^{(s)} = -\sum_{i=1}^{N} \ln p_{GM}(z_i; \mathbf{v}^{(s)})/N$ is an indicator of the cross-entropy for the sth step, and $\mathbf{v}^{(s)}$ denotes the parameters of the GM for that step.

Note that if one already knows the PDF of a nonlinear response, step 1) of the aforementioned algorithm can be replaced by a random generation of N samples from the response PDF.

3.1.1. Remark 1: Selecting sample points

One issue in selecting sample points in the aforementioned algorithm is that the nonlinear response takes a certain amount of time to achieve stationarity, thus using the whole time series including a nonstationary part will introduce errors to the estimated PDF. To reduce this error, for each of the M response histories obtained from the first step of the algorithm, we need to select $\bar{N}$ stationary response values as the sample points.

Here we provide a method to crudely estimate the time that the system would take to achieve stationarity. To begin with, the standard deviation of the response at a sequence of time points, denoted as $\text{std}[Z(j\Delta t)]$, in which $j = 1, 2, \ldots$ and Δt is the time step of the nonlinear analysis, is estimated using the recorded M response histories, and then a sigmoid function expressed as

$$f_{fit}(j) = \frac{1}{1 + e^{-aj\Delta t + b}} \tag{12}$$

is employed to fit the $\text{std}[Z(j\Delta t)]$ curve. Note that $f_{fit}(\cdot) \in (0, 1)$, thus the $\text{std}[Z(j\Delta t)]$ curve should be scaled by a factor $J/\sum_{j=1}^{J}\text{std}[Z(j\Delta t)]$ ($J\Delta t$ is the duration of the excitation) so that it approximately ranges from 0 to 1. The parameters a and b in Eq. (12) can be determined from a least-square regression analysis. A typical scaled $\text{std}[Z(j\Delta t)]$ curve and its corresponding fitting function $f_{fit}(\cdot)$ is illustrated in Fig. 3. With $f_{fit}(t)$ available, the time the system takes to achieve stationarity, denoted by $j_{ns}\Delta t$, can be estimated via

$$j_{ns} = argmin\{j | 1 - f(j\Delta t) \leqslant Tol, j = 1, 2, \ldots\} \tag{13}$$

where Tol denotes a specified tolerance. With j_{ns} determined, for each of the M response histories, $\bar{N} = J - j_{ns}$ time points corresponding to the stationary responses are selected to be the sample points, and the total number of sample points is $N = M \cdot \bar{N} = M \cdot (J - j_{ns})$.

3.1.2. Remark 2: Determining K

The optimal value of K in principle should be dependent on the specific problem being studied. However, for the one-dimensional PDF of the nonlinear response considered in this study, we have found in most cases a $K \geqslant 20$ value is sufficient to provide an

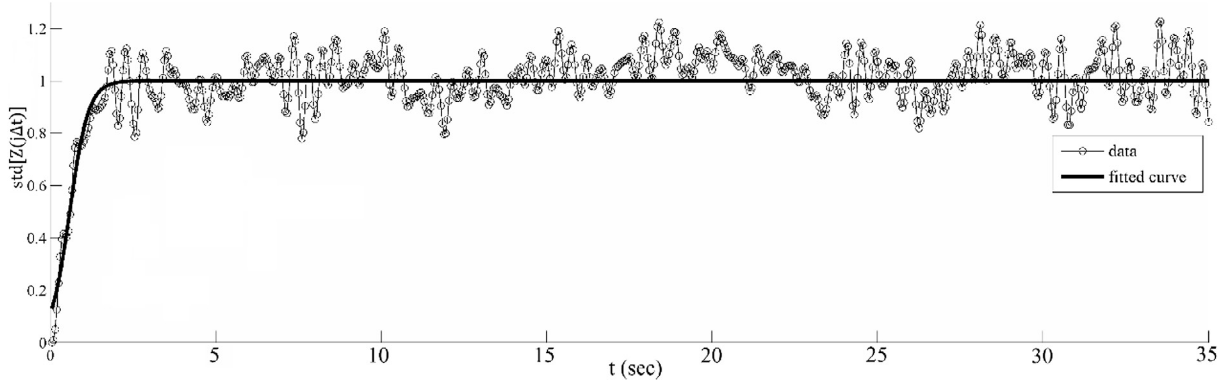

Fig. 3. A typical scaled std$[Z(j\Delta t)]$ curve and the fitting function.

accurate approximation, and the accuracy is not overly sensitive to the K value. Thus one could simply set, e.g. $K = 20$, for most practical applications. A less ad-hoc way to determine K is to run the aforementioned algorithm of GM parameter identification, starting from $K = 1$, to record the cross entropy term $\mathrm{ce}(K) = -\sum_{i=1}^{N} lnp_{GM}(z_i; \mathbf{v}^{(K)})/N$, and then K is increased by 1 to compute the corresponding cross entropy term. This procedure is continued until one of the following stopping criteria is satisfied: 1) a local minimum of $\mathrm{ce}(K)$ is achieved, i.e. $\mathrm{ce(K)} < ce(K+1), \mathrm{ce(K)} < ce(K-1)$, for $K > 1$; and 2) the variation of $\mathrm{ce}(K)$ with K is small, i.e. $\left|\frac{\mathrm{ce}(K)-\mathrm{ce}(K-1)}{\mathrm{ce}(K)}\right| < Tol$, for $K > 1$. Note that the same set of samples is used in the aforementioned procedure for various K, thus the additional computational demand of the iterative procedure is trivial.

3.1.3. Remark 3: Symmetric distributions

If a structure has a symmetric nonlinear behavior associated with response $Z(t)$, and is subjected to zero-mean stochastic excitation, the distribution $p(z)$ will be zero-mean and symmetric. Therefore, it would be ideal if the mean of the GM, $\mu_{GM} = \sum_{k=1}^{N} \alpha_k \mu_k$, is zero and the parameters of GM are symmetric. To enforce this symmetric properties being met by the GM model, one could partition the Gaussian densities into $K/2$ pairs (given K is even), and the parameters of the GM have the form $\mathbf{v} = \left\{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_{K/2}, \bar{\alpha}_{K/2}, \mu_1, \bar{\mu}_1, \ldots, \mu_{K/2}, \bar{\mu}_{K/2}, \sigma_1, \bar{\sigma}_1, \ldots, \sigma_{K/2}, \bar{\sigma}_{K/2}\right\}$, in which $\alpha_k = \bar{\alpha}_k$, $\mu_k = -\bar{\mu}_k$, and $\sigma_k = \bar{\sigma}_k$ for $\forall k$. During each step of the parameter updating process, the $K/2$ pairs of parameters are still treated as K independent ones so that they are updated in the same manner as described in the aforementioned algorithm. After $\mathbf{v}$ is updated, the values of the $\{\alpha_k, \bar{\alpha}_k\}$ and $\{\sigma_k, \bar{\sigma}_k\}$ pairs are respectively modified to their mean values, and the values for the $\{\mu_k, \bar{\mu}_k\}$ pair are modified to $\{(\mu_k - \bar{\mu}_k)/2, (-\mu_k + \bar{\mu}_k)/2\}$. This technique should be performed in conjunction with a symmetric initial parameters setting. Using this technique, when the algorithm converges the GM model is ensured to have zero mean and symmetric components. This procedure can be easily extended to the case when K is odd by introducing an additional zero-mean Gaussian density, and keeping the mean of that Gaussian density fixed to zero during the entire parameter updating process.

3.2. Identifying linear systems from the Gaussian mixture

As introduced in Section 2.1, the root-mean-square response of the kth, $k = 1, \ldots, K$, linear system is equal to σ_k, i.e. the standard deviation of the kth Gaussian density, when subjected to the zero-mean component of the stochastic excitation. From theories of linear random vibration analysis [5],

$$\sigma_k^2 = \int_{-\infty}^{\infty} |H_k(\omega)|^2 S_f(\omega) d\omega, \quad k = 1, \ldots, K \tag{14}$$

in which $H_k(\omega)$ is the frequency response function (FRF) of the linear system associated with the kth Gaussian density, and $S_f(\omega)$ is the auto power spectrum density (auto-PSD) of the zero-mean component of the excitation.

We let the linear system associated with each Gaussian density be an SDOF oscillator, and the FRF of the SDOF oscillator [16] is expressed as

$$H_k(\omega) = \frac{s_{eq,k}}{k_{eq,k} + i\omega c_{eq,k} - m_{eq,k}\omega^2}, \quad k = 1, \ldots, K \tag{15}$$

in which $k_{eq,k}$, $c_{eq,k}$ and $m_{eq,k}$ are the stiffness, damping and mass of the kth linear oscillator, respectively, and $s_{eq,k}$ is a scaling factor which depends on the response quantity of interest.

It is impossible to identify all parameters of the FRF in Eq. (15) by only using Eq. (14). To facilitate convenient and practical applications of GM-ELM, we let $m_{eq,k}$, $c_{eq,k}$ and $s_{eq,k}$ be independent of k, and set them to pre-specified values, so that only $k_{eq,k}$ needs to be identified from Eq. (14). For applications to a response of MDOF systems, in this study, m_{eq}, c_{eq} and s_{eq} are obtained by

$$\begin{aligned} m_{eq} &= \boldsymbol{\psi}^T \boldsymbol{M} \boldsymbol{\psi} \\ c_{eq} &= \boldsymbol{\psi}^T \boldsymbol{C}_0 \boldsymbol{\psi} \\ s_{eq} &= (\boldsymbol{q}^T \boldsymbol{\psi}) \cdot (\boldsymbol{\psi}^T \boldsymbol{\mathcal{F}}) \end{aligned} \tag{16}$$

where $\boldsymbol{M}$ and $\boldsymbol{C}_0$ are the mass and initial damping matrix of the nonlinear system, $\boldsymbol{\mathcal{F}}$ is the spatial distribution of the input excitation, $\boldsymbol{q}$ is a deterministic vector which depends on the response quantity of interest, and $\boldsymbol{\psi}$ is a 'representative' response shape vector. The shape vector $\boldsymbol{\psi}$ can be simply selected as the modal vector (for a linear system with the initial structural properties of the nonlinear system) which contributes the most to the response quantity of interest. If the initial structural properties would lead to unsuitable linear systems (e.g. systems with zero stiffness), one could use the equivalent linear system obtained from conventional ELM to set $\boldsymbol{\psi}$, m_{eq}, c_{eq}, and s_{eq} values. With m_{eq}, c_{eq} and s_{eq} obtained from Eq. (16), Eq. (15) is substituted into Eq. (14) and the stiffness $k_{eq,k}$ can be easily found.

4. Applications of GM-ELM

4.1. Random vibration analysis

The linear system obtained using the GM model can be used to estimate response statistics of interest. First, the instantaneous CDF and root-mean-square response of the nonlinear system can be computed using Eqs. (4) and (6), respectively.

For the up-crossing rate estimation, consider the expression [17]

$$\nu^{+}(z)=\lim_{\delta t\to 0}\frac{\Pr\{Z(t)<z\cap Z(t+\delta t)>z\}}{\delta t} \tag{17}$$

At time point t, the nonlinear system is represented by one of the K linear systems, for an infinitesimal δt it is unlikely that the linear system switches from one to another, thus Eq. (17) can be rewritten as

$$\nu^{+}(z)=\lim_{\delta t\to 0}\frac{\sum_{k=1}^{K}\alpha_k\Pr\{Z_k(t)<z\cap Z_k(t+\delta t)>z\}}{\delta t}=\sum_{k=1}^{K}\alpha_k\nu_k^{+}(z) \tag{18}$$

in which $Z_k(t)=D_k(t)+\mu_k$ denotes the response of the kth linear system, and $\nu_k^{+}(z)$ denotes the up-crossing rate of the kth linear system. One can use well-known linear random vibration solutions to estimate $\nu_k^{+}(z)$ in Eq. (18). For example, $\nu_k^{+}(z)$ can be estimated by [18]

$$\nu_k^{+}(z)=\frac{1}{2\pi}\sqrt{\frac{\lambda_{2,k}}{\lambda_{0,k}}}\exp\left[-\frac{0.5(z-\mu_k)^2}{\lambda_{0,k}}\right] \tag{19}$$

Note the presence of μ_k in Eq. (19). The spectral moment $\lambda_{j,k}$ in Eq. (19) is expressed as [5]

$$\lambda_{j,k}=\int_{-\infty}^{\infty}|\omega|^{j}|H_k(\omega)|^2S_f(\omega)d\omega \quad j=0,1,2 \tag{20}$$

where $H_k(\omega)$ is the FRF of the kth linear system identified by the procedure in Section 3.2, and $S_f(\omega)$ is the auto-PSD of the zero-mean component of the excitation.

Assuming that the up-crossings follow a Poisson process with rate $\nu^{+}(z)$, the first-passage probability for a time period T_d, $\Pr[\max Z(t)>z]_{t\in T_d}$ can be estimated as

$$\begin{aligned}\Pr[\max Z(t)>z]_{t\in T_d}&=1-\exp[-\nu^{+}(z)T_d]\\&=1-\exp\left[-\sum_{k=1}^{K}\alpha_k\nu_k^{+}(z)T_d\right]\end{aligned} \tag{21}$$

Note that the Poisson process approximation normally works well when the mean down-crossing rate is small and the process is not narrow band. In future research, the proposed method will be further developed for applications to broader class of first-passage problems.

4.2. Seismic response spectrum analysis in earthquake engineering

In some practical applications, earthquake engineering in particular, one is often interested in the mean peak absolute response of a nonlinear system over a time period T_d, denoted as $E[\max|Z(t)|]_{t\in T_d}$. Following the conventional procedure for deriving the seismic response spectrum formula [19], within the GM-ELM framework it is easy to show that $E[\max|Z(t)|]_{t\in T_d}$ can be expressed by

$$\begin{aligned}E[\max|Z(t)|]_{t\in T_d}&\cong p_z\sigma_z=\left[\sum_{k=1}^{K}\alpha_k p_z^2(\sigma_k^2+\mu_k^2)\right]^{1/2}\\&=\left[\sum_{k=1}^{K}\alpha_k\left(\frac{p_z^2}{p_k^2}D_{k,\max}^2+p_z^2\mu_k^2\right)\right]^{1/2}\\&\cong\left[\sum_{k=1}^{K}\alpha_k\left(D_{k,\max}^2+p_z^2\mu_k^2\right)\right]^{1/2}\end{aligned} \tag{22}$$

in which p_z and p_k are peak factors, the last term assumes $p_z^2/p_k^2\cong 1$, and $D_{k,\max}$ is the mean peak deformation for the kth linear system (see Fig. 1) and $D_{k,\max}$ is directly related to the seismic response spectrum. Eq. (22) inevitably involves the peak factor, this makes the formula practically not useful.

It is seen from the aforementioned derivation that the conventional approach cannot lead to a practically useful response spectrum formula for GM-ELM. In this paper a heuristic response spectrum formula will be developed using physical interpretations of GM-ELM. First, the mean peak response for each linear system in GM-ELM can be expressed by (see Fig. 1)

$$E[\max|Z_k(t)|]_{t\in T_d}=D_{k,\max}+|\mu_k|=\Gamma\cdot S_d(\omega_k,\xi_k)+|\mu_k| \tag{23}$$

in which ω_k and ξ_k respectively denote the natural frequency and damping ratio of the kth linear system, $S_d(\omega_k,\xi_k)$ denotes the ordinate of the displacement response spectrum at ω_k and ξ_k, and $\Gamma=|s_{eq}|/m_{eq}$ is a scaling factor of $S_d(\omega_k,\xi_k)$. In earthquake engineering, the displacement response spectrum ordinate $S_d(\omega_k,\xi_k)$ indicates the mean peak displacement of a linear oscillator with natural frequency ω_k and damping ratio ξ_k subjected to a specified stochastic ground motion [16]. Next, it is conjectured that the mean peak nonlinear response of interest, $E[\max|Z(t)|]_{t\in T_d}$, can be written in the form

$$E[\max|Z(t)|]_{t\in T_d}\cong\sum_{k=1}^{K}c_kE[\max|Z_k(t)|]_{t\in T_d} \tag{24}$$

in which the unknown weight c_k satisfies $c_k\in[0,1]$ and $\sum_{k=1}^{K}c_k=1$. If one lets $c_k=\alpha_k$, Eq. (24) may only provide a lower bound for $E[\max|Z(t)|]_{t\in T_d}$, since the linear system with larger mean peak response tends to contribute more to the total mean peak response than the instantaneous weight α_k indicated. Thus, $E[\max|Z(t)|]_{t\in T_d}$ lies in the range

$$\begin{aligned}\sum_{k=1}^{K}\alpha_kE[\max|Z_k(t)|]_{t\in T_d}&\leqslant E[\max|Z(t)|]_{t\in T_d}\\&\leqslant\max\left\{E[\max|Z_k(t)|]_{t\in T_d}\right\}\end{aligned} \tag{25}$$

Using Eqs. (24) and (25), it is reasonable to approximate $E[\max|Z(t)|]_{t\in T_d}$ by

$$\begin{aligned}E[\max|Z(t)|]_{t\in T_d}&\cong\sum_{k=1}^{K}\frac{l_k\alpha_k}{\sum_{k=1}^{K}l_k\alpha_k}E[\max|Z_k(t)|]_{t\in T_d}\\&=\sum_{k=1}^{K}\frac{l_k\alpha_k}{\sum_{k=1}^{K}l_k\alpha_k}[\Gamma\cdot S_d(\omega_k,\xi_k)+|\mu_k|]\end{aligned} \tag{26}$$

in which Γ is defined after Eq. (23), l_k is a binary function, l_k gives 1 if $E[\max|Z_k(t)|]_{t\in T_d}$ satisfies

$$E[\max|Z_k(t)|]_{t\in T_d}\in\left[0.95\sum_{k=1}^{K}\alpha_kE[\max|Z_k(t)|]_{t\in T_d},\max\{E[\max|Z_k(t)|]_{t\in T_d}\}\right] \tag{27}$$

and l_k gives 0 otherwise. Note that a heuristic scaling factor of 0.95 is introduced in Eq. (27) to relax the lower bound.

Eq. (26) is potentially useful in earthquake engineering practice since it provides a way to utilize existing elastic response spectra to analyze nonlinear structures subjected to earthquakes, i.e. without developing inelastic spectra for specific types of structures each time [20,21].

5. Additional remarks on GM-ELM

5.1. Understandings and possible misunderstandings of GM-ELM

One obstacle in understanding the proposed method is that the identified linear components established in GM-ELM have a non-physical behavior, i.e. the linear components can switch from one to another from time to time. A fundamental idea in the development of GM-ELM is that the 'behavior' of the linear components is presumed mainly out of mathematical considerations, rather than out of physical ones. GM-ELM is not a method that tries to establish a physical linear system that could mimic time histories of the original nonlinear system. Instead, GM-ELM establishes a set of linear systems and a reasonable way (e.g. probabilistic combination rules) to process them, so that the nonlinear response PDF/CDF is mimicked, and response statistics such as the crossing rate and first-passage probabilities can be estimated.

Since the linear system in GM-ELM lacks physical validity (consider the random switching behavior), they are not named 'equivalent linear system' in this paper. However, it is the relaxation of physical validity that provides GM-ELM addition flexibility in modelling non-Gaussian distribution behavior of nonlinear responses. Sometimes the 'physical' aspect of ELM becomes a confinement for the improvement of the accuracy. A similar philosophy is observed in TELM [10], in which the linear system makes sense in the context of first-order reliability method.

A possible misunderstanding of GM-ELM is that since the estimation of GM model requires knowledge of the PDF of the nonlinear system response, it seems GM-ELM cannot provide extra information during the linearization process. To avoid this misunderstanding, one needs to notice the fact that even if the exact PDF of a structural response is given, response statistics such as the crossing rate, first-passage probability and mean peak response over a period of time are still not available. In fact, those response statistics require information on the temporal correlation of the nonlinear structural responses, and the linear system established by GM-ELM can provide such information. Moreover, note that similar to GM-ELM, essentially TELM [10] also uses the response probability distribution (computed via FORM) to establish an equivalent linear system.

5.2. Applicability of GM-ELM

In general, GM-ELM is applicable to MDOF nonlinear structural systems subjected to Gaussian excitations or non-Gaussian excitation which can be described by a nonlinear filter and Gaussian excitations. The Gaussian excitation precondition is necessary because the response of a linear system to Gaussian excitation is also Gaussian, so that a Gaussian mixture decomposition of the nonlinear response PDF is meaningful and can be naturally related to linear systems. The constitutive law of the nonlinear system in GM-ELM does not have to be smooth. So far, the application of GM-ELM to nonlinear systems with multiple stochastic excitations, and systems with uncertain structural properties have not yet been studied. However, it seems there is no fundamental barrier that hinders the extension of GM-ELM to those problems, since the GM model is fully capable of representing multi-mode PDFs [11].

6. Numerical investigations

6.1. SDOF nonlinear oscillator

Consider a cubic oscillator governed by the differential equation

$$\ddot{Z}(t) + \dot{Z}(t) + Z^3(t) = f(t) \tag{28}$$

where $f(t)$ is a white noise process with auto-PSD $S_f(\omega) = 1/\pi$. The duration of the excitation is assumed to be 35 s. Fig. 4 shows a typical force-deformation curve of the nonlinear system.

The closed-form solution for the PDF of the nonlinear response $Z(t)$ is [2]

$$p(z) = \frac{\exp[-z^4/4]}{\int_{-\infty}^{\infty} \exp[-z^4/4]dz} \tag{29}$$

and its variance is

$$E[Z^2] = \int_{-\infty}^{\infty} z^2 p(z)dz = 0.6760 \tag{30}$$

The proposed GM-ELM with 20 Gaussian densities is used to estimate various response statistics of this example. Since the nonlinear response PDF is given by Eq. (29), dynamic analysis is not required in GM-ELM. In terms of Eq. (28), for this SDOF problem m_{eq}, c_{eq} and s_{eq} in Eq. (16) are all set to 1. The results of GM-ELM are obtained using 1.0×10^5 samples directly drawn from Eq. (29). Note that since the GM-ELM depends on random samples of response values to identify the GM model, the final GM-ELM analysis results will fluctuate slightly. In this and the following example, typical results of GM-ELM are reported to illustrate the accuracy one could expect from the method.

The variance of the displacement obtained by the conventional ELM is 0.5540 m^2, which is 18.05% smaller than the exact solution, while the result obtained from GM-ELM is 0.6738 m^2, which is only 0.33% smaller than the exact solution. This result is expected since GM-ELM utilizes samples drawn from the exact PDF.

The mean up-crossing rates and first-passage probabilities obtained from the conventional ELM (using MCS with the equivalent linear system), GM-ELM (using Eqs. (18), (19) and (21)) and MCS with 1.0×10^5 samples are illustrated in Fig. 5. It is observed that GM-ELM provides accurate estimates on both the crossing rate and the first-passage probability and the error increases as the threshold level increases. It is important to note that the PDF expressed by Eq. (29) provides no information on the temporal correlation of the nonlinear response, so that crossing rate and first-passage probability of the nonlinear response cannot be estimated from the PDF. However, the results in Fig. 5 indicate that by using the response PDF in conjunction with some structural properties (m_{eq}, c_{eq} and s_{eq} in Eq. (15)), the linear system (the 'extra' information) generated in GM-ELM leads to an accurate estimation of crossing rate and first-passage probability. This observation further

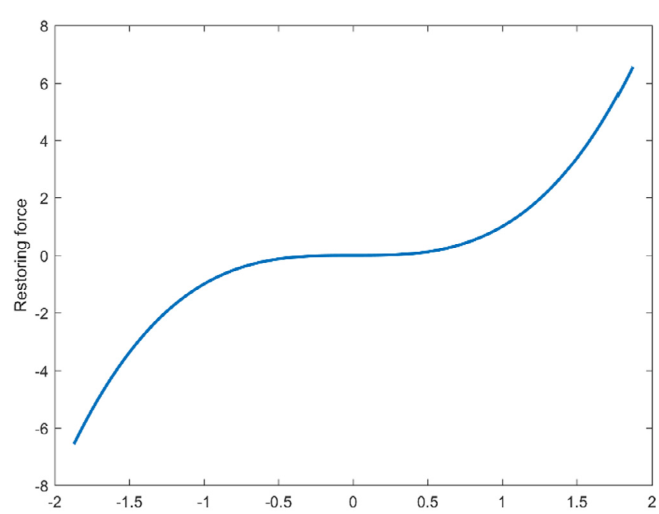

Fig. 4. Force-deformation curve of the cubic oscillator.

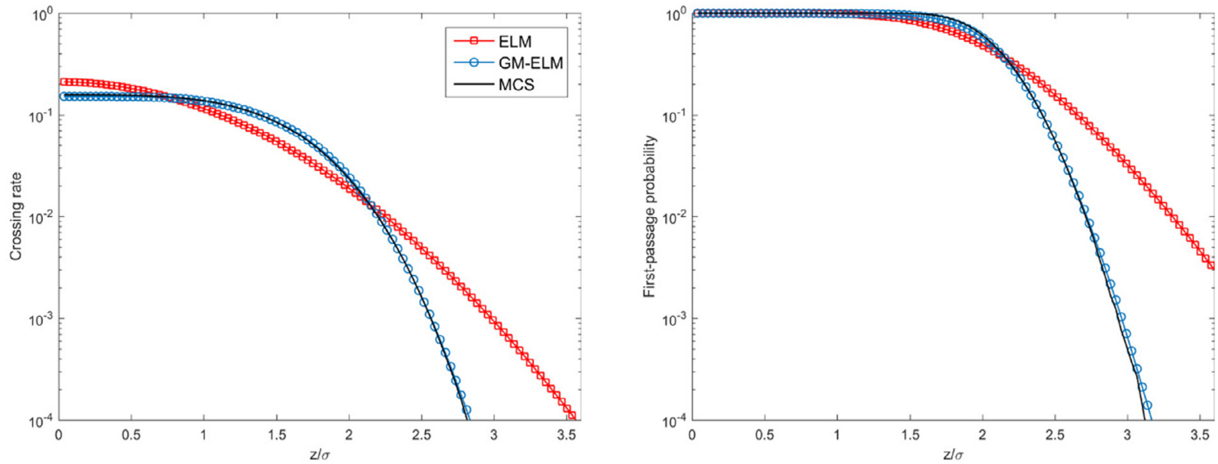

Fig. 5. Mean up-crossing rates (left) and first-passage probabilities (right) obtained from various methods.

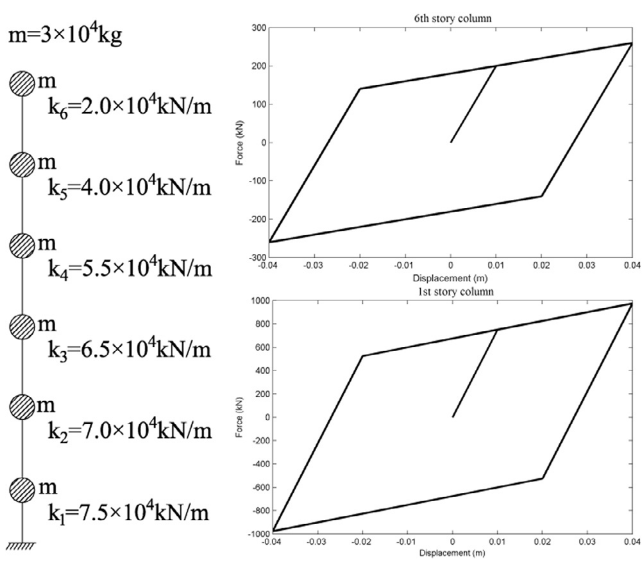

Fig. 6. 6-DOF shear-building model.

responds to the misunderstandings of GM-ELM mentioned in Section 5.1.

Finally, the mean peak absolute responses estimated from ELM (using MCS with the equivalent linear system), GM-ELM (using the response spectrum formula Eq. (26)) and MCS are 2.0343 m $(11.3\%\ \text{error})$, 1.8209 m$(-0.33\%\ \text{error})$ and 1.8270 m, respectively. In this and the following example the response spectrum ordinates of Eq. (26) are computed via MCS, while in earthquake engineering practice the 'design spectrum' is often directly available from seismic codes.

It is seen that for all response statistics considered in this example, the proposed GM-ELM shows superior accuracy compared to ELM.

6.2. MDOF nonlinear system with non-smooth hysteretic models

Consider a 6-DOF shear-building model shown in Fig. 6. The force-deformation behavior of each column is described by a bilinear hysteretic model shown in the figure. The yield deformation of each story is set to 0.01 m. The structure has an initial fundamental period of 0.576 s and the second mode period of 0.238 s. Rayleigh damping with 5% damping ratio in modes 1 and 2 is assumed. The building is subjected to a stochastic ground motion with the auto-PSD described by a modified Kanai-Tajimi model suggested by Clough and Penzien [16]

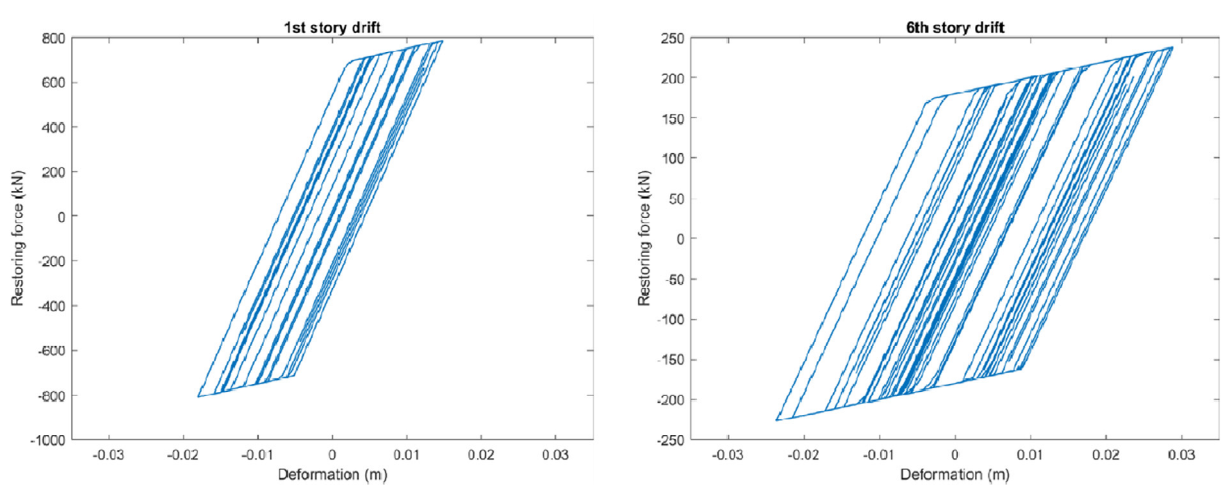

Fig. 7. Typical hysteretic loops for 1st and 6th column.

$$S_f(\omega) = S_0 \frac{\omega_f^4 + 4\zeta_f^2\omega_f^2\omega^2}{(\omega_f^2 - \omega^2)^2 + 4\zeta_f^2\omega_f^2\omega^2} \frac{1}{(\omega_s^2 - \omega^2)^2 + 4\zeta_s^2\omega_s^2\omega^2} \tag{31}$$

where $S_0 = 0.016\ \mathrm{m}^2/\mathrm{s}^3$ is a scale factor, $\omega_f = 15$ rad/s and $\zeta_f = 0.6$ are the filter parameters representing, respectively, the natural frequency and damping ratio of the soil layer, and $\omega_s = 1.5$ rad/s and $\zeta_s = 0.6$ are parameters of a second filter that is introduced to assure finite variance of the ground displacement. The duration of the ground motion is assumed to be 40 s.

The results of GM-ELM with 20 Gaussian densities are obtained using response samples obtained from 29 runs of dynamic analy-

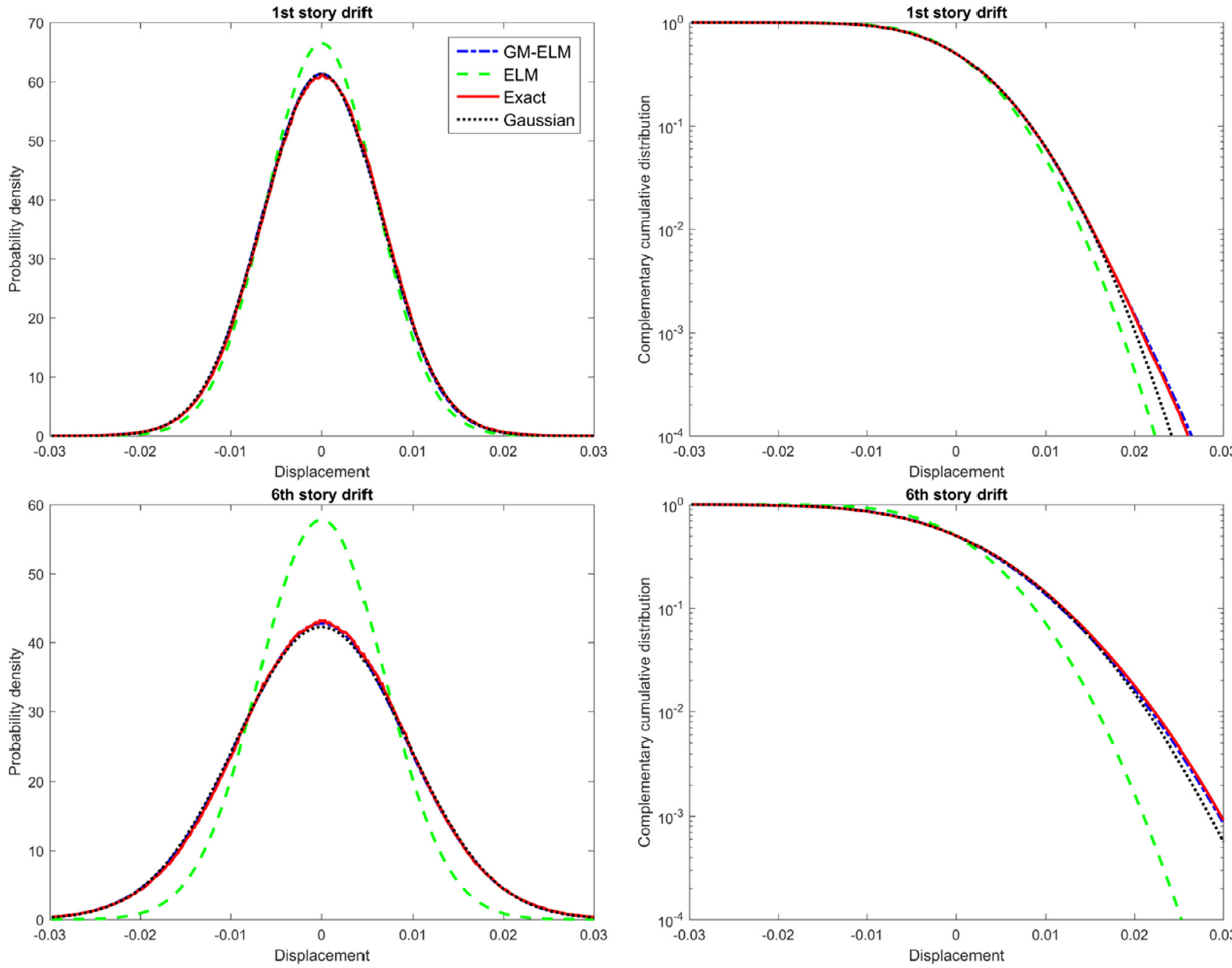

Fig. 8. PDF and complementary CDF estimations of the MDOF system by GM-ELM and ELM.

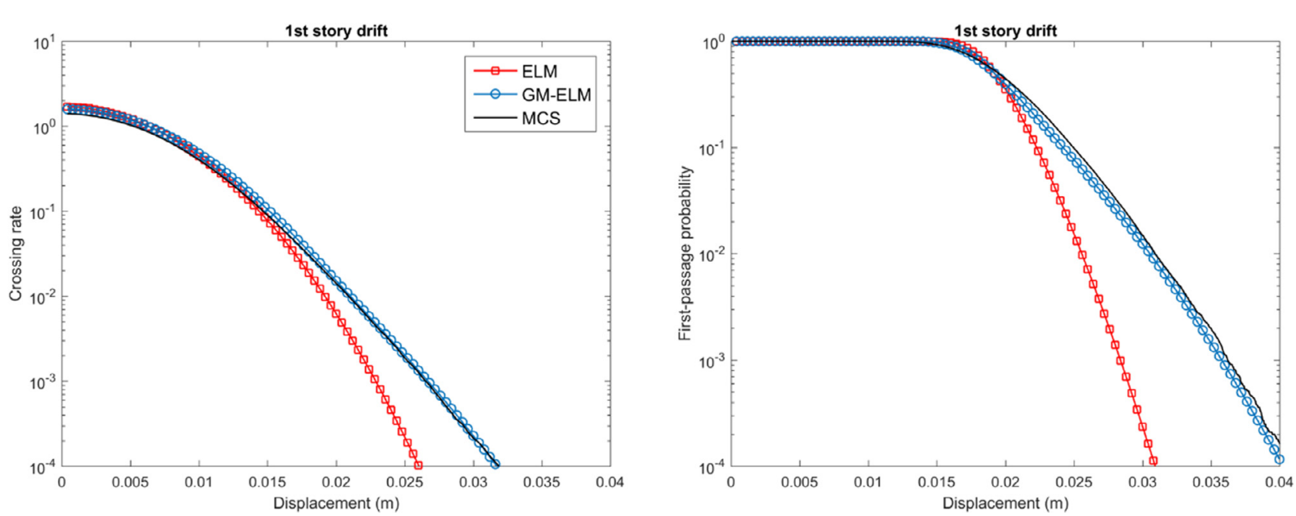

Fig. 9. Mean up-crossing rates and first-passage probabilities for the 1st story drift.

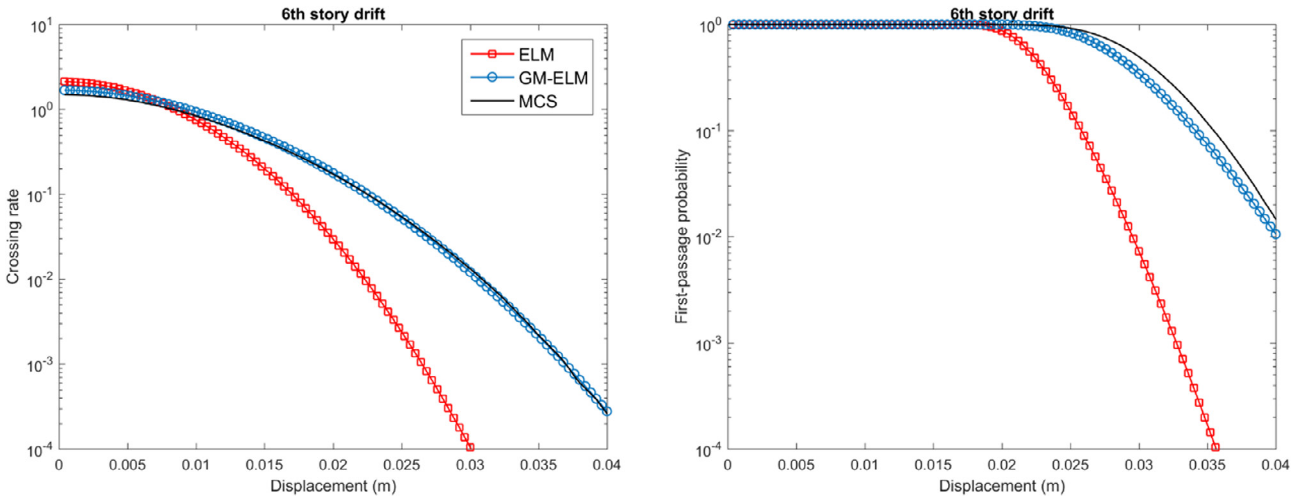

Fig. 10. Mean up-crossing rates and first-passage probabilities for the 6th story drift.

sis. Typical hysteretic loops for the 1st and 6th column subjected to the stochastic ground motion are illustrated in Fig. 7.

The PDFs and complementary CDFs for the 1st and 6th story drift obtained from GM-ELM, ELM and MCS (using histogram) are illustrated in Fig. 8. Also Gaussian distributions with the variance equal to the variance of the nonlinear response are shown in Fig. 8, to illustrate the non-Gaussianity of the nonlinear response. It is seen from Fig. 8 that results obtained from GM-ELM are noticeably more accurate than the results obtained from ELM (especially for the 6th story drift). Note that m_{eq}, c_{eq} and s_{eq} in Eq. (16) are computed using the first modal vector for the 1st story drift, and the second modal vector for the 6th story drift.

Figs. 9 and 10 respectively show the mean up-crossing rates and first-passage probabilities for the 1st and 6th story drift obtained from ELM, GM-ELM and MCS using 1.0×10^5 samples.

The mean peak absolute drift for the 1st story estimated from ELM (using MCS for the equivalent linear system), GM-ELM (using the response spectrum formula Eq. (26)) and MCS are 0.0185 m (-8.42% error), 0.0203 m (0.50% error) and 0.0202 m, respectively, while the results for the 6th story are 0.0219m (-27.72% error), 0.0316 m (4.29% error) and 0.0303 m, respectively.

Similar to the previous example, it is seen that for all response statistics considered in this example, the proposed GM-ELM is significantly more accurate than ELM.

7. Conclusions

A new equivalent linearization method based on Gaussian mixture (GM) distribution model is developed for random vibration analysis of nonlinear structural systems. Due to properties of the GM distribution model, the GM based equivalent linearization method (GM-ELM) can decompose the non-Gaussian response of a nonlinear system into multiple Gaussian responses of linear single-degree-of-freedom (SDOF) oscillators. A simple method to identify parameters of the linear SDOF oscillators is proposed. Using the linear system of GM-ELM, methods to compute response statistics as the mean up-crossing rate and first-passage probability of the nonlinear system are developed. A response spectrum formula is also proposed to compute the mean peak response of the nonlinear system using elastic response spectra.

GM-ELM is illustrated and tested by two numerical examples. In the first example of a cubic SDOF oscillator subjected to white noise excitation, the analysis results indicate that in contrast with the conventional ELM, GM-ELM accurately captures the non-Gaussianity of the nonlinear response. For all response statistics considered, GM-ELM shows superior accuracy compared to ELM. The second example is a 6-DOF shear-building model that has bilinear hysteretic force-deformation relation for the lateral load-carrying mechanism of each story. The building is subjected to a stochastic ground motion described by a modified Kanai-Tajimi model. Similar to the first example, the second example also confirms the superior accuracy of the proposed GM-ELM compared to ELM. The idea of using linear elastic spectra in conjunction with the GM-ELM is tested successfully as well. This idea will be further developed and thoroughly tested in future studies.

So far GM-ELM has not yet been extended to problems with multiple stochastic excitations, e.g. structures under multi-component or/and multiple-support excitations. To make GM-ELM work for such problems, the procedure to identify linear systems needs to be modified. Studies on this topic are currently ongoing. Also, a thorough comparison between GM-ELM and tail-equivalent linearization method (TELM) in the context of applications in earthquake engineering is currently underway.

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References

[1] Caughey TK. Equivalent linearization techniques. J Acoust Soc Am 1963;35:1706–11.
[2] Atalik TS, Utku S. Stochastic linearization of multi-degree of freedom nonlinear systems. Earthquake Eng Struct Dynam 1976;4:411–20.
[3] Wen YK. Equivalent linearization for hysteretic systems under random excitation. J Appl Mech (ASME) 1980;47:150–4.

[4] Casciati F, Faravelli L, Hasofer AM. A new philosophy for stochastic equivalent linearization. Probab Eng Mech 1993;8(3–4):179–85.
[5] Lutes LD, Sarkani S. Random vibration: analysis of structural and mechanical systems. Burlington, MA: Elsevier Butterworth Heinemann; 2004.
[6] Rubinstein RY, Melamed B. Modern simulation and modeling. New York, NY: Wiley; 1998.
[7] Elishakoff I, Zhang R. Comparison of new stochastic linearization criteria. In: Bellomo N, Casciati F, editors. Nonlinear stochastic mechanics. Berlin: Springer-Verlag; 1992. p. 201–12.
[8] Ditlevsen O, Madsen HO. Structural reliability methods. Wiley; 1999.
[9] Der Kiureghian A. First- and second-order reliability methods. In: Nikolaidis E, Ghiocel DM, Singhal S, editors. Engineering design reliability handbook. Boca Raton (FL): CRC Press; 2005. Chapter 14.
[10] Fujimura K, Der Kiureghian A. Tail-equivalent linearization method for nonlinear random vibration. Probab Eng Mech 2007;22:63–76.
[11] McLachlan G, Peel D. Finite mixture models. New York: Wiley; 2000.
[12] Rahman S, Xu H. A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics. Probab Eng Mech 2004;19:393–408.
[13] Schöbi R, Sudret B, Wiart J. Polynomial-chaos-based Kriging. Int J Uncertainty Quantif 2015;5(2):171–93.
[14] Wang Z, Broccardo M, Der Kiureghian A. An algorithm for finding a sequence of design points in reliability analysis. Struct Saf 2016;58:52–9.
[15] Kullback S, Leibler RA. On information and sufficiency. Ann Math Stat 1951;1:79–86.
[16] Clough RW, Penzien J. Dynamic of structures. New York: McGraw-Hill; 1975.
[17] Hagen O, Tvedt L. Vector process out-crossing as parallel system sensitivity measure. J Eng Mech, ASCE 1991;117(10):2201–20.
[18] Rice OC. Mathematical analysis of random noise. Bell Syst Tech J 1944;24:46–56.
[19] Wilson EL, Der Kiureghian A, Bayo EP. A replacement for the SRSS method in seismic analysis. Earthquake Eng Struct Dynam 1981;9(2):187–94.
[20] Chopra AK. Dynamics of structures: theory and applications to earthquake engineering. Prentice-Hall; 2001.
[21] Building seismic safety council. Guidelines for seismic rehabilitation of buildings. Washington DC: FEMA 356; 2003.